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Radiation reaction of a classical quasi-rigid extended particle

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Abstract

The problem of the self-interaction of a quasi-rigid classical particle with an arbitrary spherically symmetric charge distribution is completely solved up to the first order in the acceleration. No ad hoc assumptions are made. The relativistic equations of conservation of energy and momentum in a continuous medium are used. The electromagnetic fields are calculated in the reference frame of instantaneous rest using the Coulomb gauge; in this way the troublesome power expansion is avoided. Most of the puzzles that this problem has aroused are due to the inertia of the negative pressure that equilibrates the electrostatic repulsion inside the particle. The effective mass of this pressure is $-U_{\mathrm{e}}/(3c^2)$, where U_{e} is the electrostatic energy. When the pressure mass is taken into account the dressed mass m turns out to be the bare mass plus the electrostatic mass $m = m_0 + U_{\mathrm{e}}/c^2$. It is shown that a proper mechanical behaviour requires that $m_0 > U_{\mathrm{e}}/3c^2$. This condition poses a lower bound on the radius that a particle of a given bare mass and charge may have. The violation of this condition is the reason why the Lorentz–Abraham–Dirac formula for the radiation reaction of a point charge predicts unphysical motions that run away or violate causality. Provided the mass condition is met the solutions of the exact equation of motion never run away and conform to causality and conservation of energy and momentum. When the radius is much smaller than the wavelength of the radiated fields, but the mass condition is still met, the exact expression reduces to the formula that Rohrlich (2002 *Phys. Lett.* A **303** 307) has advocated for the radiation reaction of a quasi-point charge.

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1. Introduction

Perhaps the last unsolved problem of classical physics is the self-interaction of charged particles. We refer to a classical particle with a spherically symmetric charge density moving under the influence of an external force $\boldsymbol{F}_{\mathrm{ex}}$ and the force $\boldsymbol{F}_{\mathrm{em}}$ due to the electromagnetic fields that it itself generates. This last self-force can be expressed as the sum of the negative time derivative of the momentum of the local electromagnetic fields that surround the particle ($\boldsymbol{F}_{\mathrm{L}}$) plus the radiation reaction ($\boldsymbol{F}_{\mathrm{R}}$). Fairly complete relations of more than a century of history can be found in the monograph of Yaghjian [1], in the review of Rohrlich [2] and in chapter 16 of the textbook of Jackson [3]. The published solutions of this problem are filled with puzzles. Let us consider the most notorious ones. We will use Gauss units.

(1) *Spontaneous self-acceleration*. The Lorentz–Abraham–Dirac (LAD) formula for the radiation reaction of a point charge [4–6], which in the non-relativistic limit reduces to $\boldsymbol{F}_{\mathrm{R}} = \frac{2}{3}q^2\dot{\boldsymbol{a}}/c^3$, has solutions in which the particle, in the absence of an external force, spontaneously accelerates itself approaching speed c (run-away solutions).

(2) *Violation of causality*. There are also solutions of the LAD equation that do not run away, but where the acceleration depends on values that the applied force will take in the future.

(3) *Power unbalance*. The electrostatic field (i.e. the electric field that is present when the particle is not accelerated) does not contribute to the net force on the particle, because the integral over the volume of the corresponding force density vanishes. It, nevertheless, contributes to the power, because of the Lorentz contraction. As a result, the total power of electromagnetic forces differs from the product $\boldsymbol{F}_{\mathrm{em}} \cdot \boldsymbol{v}$. This was noted by Abraham in 1904 [7]. In 1905, Poincaré [8, 9] showed that the existence of a pressure was required in order to balance the electrostatic repulsion and thus ensure the stability of the particle. The work of the pressure, as the volume of the particle changes, was supposed to fix the power unbalance. But there is a problem, in [8] and in the introduction of [9] (p. 20) Poincaré refers to an *external* pressure, whose work is indeed proportional to the change of volume, but the pressure that is actually calculated (p 63 of [9]) is an *internal* negative pressure, whose work over the whole particle is obviously zero. It is worth noting that the results of Lorentz, Abraham and Poincaré on this subject conform with relativity in spite of being previous to the 1905 Einstein paper, because they assume an ad hoc Lorentz contraction.

(4) *The* $4/3$ *problem*. The electrostatic energy U_{e} should contribute to the mass of the particle, so if m_0 is the bare mass then the dressed mass must be $m = m_0 + U_{\mathrm{e}}/c^2$. But if one computes the momentum of the electromagnetic field that surrounds a spherical charge distribution moving with velocity $\boldsymbol{v}$, one obtains $\frac{4}{3}U_{\mathrm{e}}\gamma\boldsymbol{v}/c^2$. This value agrees with that part of the self-force that is proportional to the acceleration ($\boldsymbol{F}_{\mathrm{L}}$). To make matters more confusing the energy of such field is $U_{\mathrm{e}}\gamma\left(1 + \frac{1}{3}(v/c)^2\right)$, which has an additional $\frac{1}{3}(v/c)^2$ term.

Schwinger [10] has shown, for a spherically symmetrical charge distribution moving uniformly, that there are tensors which depend on the fields, that can be added to the electromagnetic energy–momentum–stress tensor $T^{\mu\nu}$ in order to obtain a divergence-less total tensor. He proposed that such 'corrected' tensors should be used for calculating the four-momentum of the fields. He found two possible tensors, corresponding to electromagnetic masses U_{e}/c^2 and $\frac{4}{3}U_{\mathrm{e}}/c^2$. Rohrlich [2] has remarked that any value of the electromagnetic mass in between is also possible.

Instead, Singal [11] has shown that the conventional theory is consistent when a proper account is taken of all the energy and momentum associated with the electromagnetic phenomenon in the system.

(5) *Apparent violation of energy and momentum conservation*. The Larmor [12] formula for the radiated power by a point charge in the instantaneous rest frame is $P_{\mathrm{rad}} = \frac{2}{3}q^2a^2/c^3$. But the LAD formula predicts that if the particle is moving with constant acceleration

the reaction force is zero and therefore also its power. From where is the radiated power coming? A similar situation occurs with the momentum. In the rest frame, the rate of radiated momentum vanishes. Where does the impulse of the radiation reaction force go? One should expect that the electromagnetic field surrounding the particle acts as a reservoir of energy and momentum.

The classical electromagnetism of a continuous current density j^μ is a perfectly well-behaved theory that conserves energy and momentum and conforms to causality. Solutions that run away or violate causality simply cannot appear. There are various sources of the problems.

The run-away solutions and the violations of causality should be attributed to the $R \to 0$ limit. We will show for any spherical charge distribution that in order to avoid run-away solutions the total effective mass of the particle m must be greater than the mass of the electromagnetic fields surrounding it, i.e. $\frac{4}{3}U_\mathrm{e}/c^2$. The reason is simple: $m - \frac{4}{3}U_\mathrm{e}/c^2$ is the effective mass of *matter*, if it were negative the acceleration would be opposite to the force. As $U_\mathrm{e} \geqslant \frac{1}{2}q^2/R$ there is an absolute minimum for the radius of a classical particle of mass m and charge q, $R \geqslant \frac{2}{3}q^2/mc^2$. Therefore, the $R \to 0$ limit is unphysical, because $R = 0$ is interior to an unphysical region.

The violation of causality is due to the expansion in powers of R that is used to evaluate the electromagnetic fields and the self-force. As the fields are linear functionals of the *retarded* currents j^μ, so must also be $\boldsymbol{F}_\mathrm{em}$. As this self-force vanishes if the particle is not accelerated, it should be a functional of the retarded acceleration $\boldsymbol{a}(t - |\boldsymbol{x} - \boldsymbol{y}|/c)$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are two points of the particle. We will find such functional for an arbitrary charge density. So, for the exact solution of an extended particle there are no pre-accelerations or violations of causality. Of course if we make an expansion in powers of $|\boldsymbol{x} - \boldsymbol{y}|$ and we take any finite number of terms we get an instantaneous result which violates causality. If we take the $R \to 0$ limit we are left with the R^{-1} and the R^0 terms; the first one diverges and corresponds to the local fields reaction $\boldsymbol{F}_\mathrm{L}$, while the second is the LAD result for the radiation reaction $\boldsymbol{F}_\mathrm{R}$; as said above such limit is unphysical. In the calculation of Dirac [6] the violation of causality is explicit, because the *advance* Green's function is used.

The exact self-force of a sphere of radius R with a uniformly charged surface has been calculated by Sommerfeld [13], Page [14], Caldirola [15] and Yaghjian [1]. Their result confirm what we have been discussing [2, 16]. Our general result yields the Sommerfeld–Page–Caldirola–Yaghjian formula for the charged spherical shell.

The power unbalance and the 4/3 problems have a common origin. They are due to having overlooked a purely relativistic effect: *pressure has inertia*. In our calculation, we use the relativistic momentum–energy conservation formula for a continuous medium. Here, the momentum–energy tensor of matter, the stress tensor and the force density appear. We have to remark that this stress tensor is not some ad hoc addition, it is just the stress tensor that any continuous medium should have. At rest the stress tensor is determined by the electrostatic repulsion and is purely spatial, but when the particle is moving the temporal components of the tensor do not vanish. Those components contribute to the momentum density and to the energy density of the particle. By integration over the volume, one obtains the equation of motion and the energy equation. The pressure contribution to the mass of the particle is $-\frac{1}{3}U_\mathrm{e}/c^2$, which is just what is needed to fix the 4/3 problem. The power unbalance also disappears.

Finally, the apparent violation of energy and momentum conservations is understood by considering all the contributions to the energy and momentum of the electromagnetic field that surrounds the particle. When the particle is not accelerated there is an electromagnetic field around the particle, the local field, which contributes to the energy and momentum of the particle. Call u_{L} and $\boldsymbol{S}_{\mathrm{L}}$ its energy density and Poynting vector. In addition to this, when the particle is accelerated there is also the radiated field with energy density u_{R} and Poynting vector $\boldsymbol{S}_{\mathrm{R}}$. But the energy density and the Poynting vector are quadratic functions of the fields, so the total energy density and Poynting vector have additional cross terms, $\boldsymbol{S} = \boldsymbol{S}_{\mathrm{L}} + \boldsymbol{S}_{\mathrm{R}} + \boldsymbol{S}_{\mathrm{C}}$ and $u = u_{\mathrm{L}} + u_{\mathrm{R}} + u_{\mathrm{C}}$. These cross-terms act as energy and momentum reservoirs.

The point charges ($R = 0$) are inconsistent with classical electrodynamics, but phenomenologically a point charge is a body with a radius that is negligible as compared to the other relevant distances, in particular to the wavelength of the fields with which it interacts. We will call such particles physical point charges, while we call mathematical point charges those with $R = 0$. We will show that the acceleration can be expressed as a time integral of the retarded applied force. When the physical point charge conditions are fulfilled the expression reduces to an equation similar to the LAD motion equation, but replacing $\dot{\boldsymbol{a}}$ by $\dot{\boldsymbol{F}}_{\mathrm{ex}}/m$, exactly as was recently proposed by Rohrlich [17].

2. Quasi-rigid motion

A perfectly rigid body is inconsistent with relativity because it implies instantaneous propagation of signals. In general, if a body is accelerated it is stressed and deformed. How this happens depends on the particular characteristics of the body and the applied force. If the external force is more or less uniform through the body and the acceleration is small and does not change too rapidly the body can, as seen in the instantaneous rest frame, continuously maintain its shape suffering a negligible deformation compared with particle radius. The bound on the acceleration $\boldsymbol{a}$ is easily determined. The speed change during the time that a signal takes to traverse the particle should be negligible with respect to the speed of sound (speed of signals in the particle). Hence, $(2R/v_s)a \ll v_s$. For a particle of maximum stiffness; $v_{\mathrm{s}} = c$ and so $a \ll c^2/2R$, which is an immense bound even for macroscopic particles. The bound of the time derivative of the acceleration is obtained by comparison of the relative rate of change of the acceleration with the inverse of the transit time, $\dot{a}/a \ll v_{\mathrm{s}}/2R$. Again assuming maximum stiffness, $\dot{a}/a \ll c/2R$. If both the conditions are satisfied, the motion of the particle is essentially rigid and the internal degrees of freedom of the particle are not excited. We will assume that this is the case. We are explicitly excluding effects produced by deformation of the particle, such as polarization.

Consider a spherically symmetric particle. Let $\boldsymbol{x}_0(t)$ be the position of the centre of the particle in the laboratory reference frame. Let $\boldsymbol{v}_0(t)$ and $\boldsymbol{a}_0(t)$ be the velocity and acceleration of that point. The instantaneous rest frame would be defined by a Lorentz transformation of velocity $\boldsymbol{v}_0$ and 'parallel' axes. Let $\boldsymbol{y}$ be the position relative to the centre of an element of the particle in the rest frame. We will assume that the motion of each element of the particle is given by the following expression:

$$\boldsymbol{x}(\boldsymbol{y}, t) = \boldsymbol{x}_0(t) + \boldsymbol{y} + \left(\gamma_0^{-1} - 1\right)(\boldsymbol{y} \cdot \hat{\boldsymbol{v}}_0)\hat{\boldsymbol{v}}_0, \tag{1}$$

where $\gamma_0 = (1 - (v_0/c)^2)^{-\frac{1}{2}}$. This expression is equivalent to the Lorentz transformation of the laboratory relative position $\boldsymbol{x}(\boldsymbol{y}, t) - \boldsymbol{x}_0(t)$ to the rest frame position $\boldsymbol{y}$. If the particle is not accelerated, (1) reproduces correctly the Lorentz contraction.

If the particle is accelerated the velocity is different at different points of the particle

$$\boldsymbol{v}(\boldsymbol{y}, t) = \frac{\partial \boldsymbol{x}(\boldsymbol{y}, t)}{\partial t} \tag{2}$$

$$= \boldsymbol{v}_0(t) + \frac{\mathrm{d}\gamma_0^{-1}}{\mathrm{d}t}(\boldsymbol{y} \cdot \hat{\boldsymbol{v}}_0)\hat{\boldsymbol{v}}_0 + \left(\gamma_0^{-1} - 1\right)\frac{\mathrm{d}}{\mathrm{d}t}[(\boldsymbol{y} \cdot \hat{\boldsymbol{v}}_0)\hat{\boldsymbol{v}}_0], \tag{3}$$

but in the rest reference frame all the points are at rest $\boldsymbol{v}(\boldsymbol{y}, t) = 0$. The accelerations at different points differ even in the rest frame, but the difference in that case is negligible $\boldsymbol{a}(\boldsymbol{y}, t) = (1 - \boldsymbol{y} \cdot \boldsymbol{a}_0/c^2)\boldsymbol{a}_0$.

It is convenient to define $\delta\boldsymbol{v}$ as the difference between the velocity of a point and the velocity of the centre

$$\delta\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{y}, t) - \boldsymbol{v}_0(t). \tag{4}$$

3. Extended particle equation

The dynamics of an extended particle is determined by the momentum–energy tensor $\Theta^{\mu\nu}$, the stress tensor $P^{\mu\nu}$ and the force density f^{μ} with the equation [18]

$$\nabla_\mu(\Theta^{\mu\nu} + P^{\mu\nu}) = f^{\nu}. \tag{5}$$

For a spherically symmetric particle, the proper density of rest mass $\mu(y)$, which is a four-scalar, is a function of the radial $y = |\boldsymbol{y}|$ coordinate in the rest frame. The integral of $\mu(y)$ over the proper volume is the bare rest mass of the particle m_0. The tensor $\Theta^{\mu\nu}$ is given by

$$\Theta^{\mu\nu} = \mu u^{\mu} u^{\nu}, \tag{6}$$

where u^{μ} is the four-velocity ($u^i = \gamma v^i$, $u^0 = \gamma c$).

The stress tensor in the rest frame is symmetric and purely spatial, $P^{0\nu} = P^{\nu 0} = 0$. We are using the metric tensor $g^{\mu\nu}$ that has a positive trace. In general, $P^{00} = v_i v_j P^{ij}/c^2$ and $P^{0i} = P^{i0} = P^{ij} v_j/c$. At rest, because of spherical symmetry, the stress tensor must be of the form $\mathbf{P} = P_r\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + P_t(\mathbf{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}})$, where $\mathbf{I}$ is the identity tensor and $\hat{\boldsymbol{r}}$ is the radial unit vector. Also at rest the force density $\boldsymbol{f}$ is the electrostatic repulsion and (5) reduces to $\nabla \cdot \mathbf{P} = \boldsymbol{f}$. This equation determines $\mathbf{P}$, but the relation between the radial stress P_{r} and the transverse stress P_{t} depends on the particular elasticity properties of the particle. Nevertheless, the final result after integrating over the volume, should be independent of the particular elasticity model. Therefore, for the sake of simplicity we will assume a Pascalian (isotropic) pressure; that is, in the rest frame $P^{ij} = P\delta_{ij}$. The pressure P is a four-scalar. Then, in a generic reference frame

$$P^{\mu\nu} = P\left(g^{\mu\nu} + \frac{u^{\mu}u^{\nu}}{c^2}\right). \tag{7}$$

The momentum and energy equations are then

$$\frac{\partial}{\partial t}\left[\left(\mu + \frac{P}{c^2}\right)\gamma^2 v\right] + \nabla \cdot \left[\left(\mu + \frac{P}{c^2}\right)\gamma^2 \boldsymbol{v}\boldsymbol{v}\right] + \nabla P = \boldsymbol{f} \tag{8}$$

and

$$\frac{\partial}{\partial t}[(\mu c^2 + P)\gamma^2 - P] + \nabla \cdot [(\mu c^2 + P)\gamma^2 \boldsymbol{v}] = \boldsymbol{f} \cdot \boldsymbol{v}. \tag{9}$$

It is interesting to compare these equations with the non-relativistic ones. The only pressure terms that appear in the non-relativistic equations are the gradient in (8) and the pressure that is inside the divergence of (9). Note that the P/c^2 terms in (8) do not disappear in the limit $v/c \to 0$. This means that, in this case, the usual non-relativistic limit, $v/c \to 0$, does not yield the actual non-relativistic result. In other words, the inertia of pressure is a purely relativistic phenomenon.

Neglecting the inertia of pressure is at the origin of some of the discussed puzzles. The P/c^2 terms that add up to μ are related to the 4/3 problem. The additional $-P$ term in the time derivative of (9) is related to the power unbalance.

In order to obtain the equations of motion for the particle, we integrate over the whole volume. As μ and P vanish outside the particle the divergence terms disappear. That is why, in the Newtonian mechanics, the pressure has no effect on the motion of the whole particle. Let us consider the first term of the left-hand side of (8),

$$\int \mathrm{d}^3x \frac{\partial}{\partial t}\left[\left(\mu+\frac{P}{c^2}\right)\gamma^2 \boldsymbol{v}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x \left(\mu+\frac{P}{c^2}\right)\gamma^2(\boldsymbol{v}_0+\delta\boldsymbol{v}) \tag{10}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma_0^2\int \mathrm{d}^3x \left(\mu+\frac{P}{c^2}\right)(\boldsymbol{v}_0+\delta\boldsymbol{v})\right] + \mathrm{O}(a^2) \tag{11}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma_0^2\int \mathrm{d}^3x \left(\mu+\frac{P}{c^2}\right)\boldsymbol{v}_0\right] + \mathrm{O}(a^2) \tag{12}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma_0\int \mathrm{d}^3y \left(\mu+\frac{P}{c^2}\right)\boldsymbol{v}_0\right] + \mathrm{O}(a^2). \tag{13}$$

In (10), we have used (4). In (11), we have replaced γ by γ_0, $\gamma = \gamma_0 + \mathrm{O}(a)$. In (11), the term $\delta\boldsymbol{v}$ vanishes because $\delta\boldsymbol{v}$ is odd in $\boldsymbol{y}$. In (13), we have transformed the integral to the rest frame, which gives a factor γ_0^{-1}. In the end we are left with

$$\frac{\mathrm{d}}{\mathrm{d}t}[(m_0+m_\mathrm{P})\gamma_0\boldsymbol{v}_0] = \boldsymbol{F}, \tag{14}$$

where we have defined the pressure mass m_P and the total force $\boldsymbol{F}$ as

$$m_\mathrm{P} = c^{-2}\int \mathrm{d}^3y\, P \tag{15}$$

and

$$\boldsymbol{F} = \int \mathrm{d}^3x\, \boldsymbol{f}. \tag{16}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(m_0+m_\mathrm{P})c^2\gamma_0 - m_\mathrm{P}c^2\gamma_0^{-1}\right] = \boldsymbol{F}\cdot\boldsymbol{v}_0 + \int \mathrm{d}^3x\, \boldsymbol{f}\cdot\delta\boldsymbol{v}. \tag{17}$$

Because of the following identity:

$$c^2\frac{\mathrm{d}\gamma_0}{\mathrm{d}t} = \boldsymbol{v}_0\cdot\frac{\mathrm{d}\gamma_0\,\boldsymbol{v}_0}{\mathrm{d}t} \tag{18}$$

if m_P is constant, it must be that

$$-m_\mathrm{P}c^2\frac{\mathrm{d}\gamma_0^{-1}}{\mathrm{d}t} = \int \mathrm{d}^3x\, \boldsymbol{f}\cdot\delta\boldsymbol{v}. \tag{19}$$

The pressure has two effects: first, it contributes to the effective mass of the particle; second, as long as (19) is verified, it cures the power unbalance. The pressure mass m_P could be negative, as it is in the charged particle case. A proper mechanical behaviour requires a positive effective mass $m_0 + m_\mathrm{P} > 0$, otherwise the acceleration would be opposite to the force.

In the rest of the paper, there will be no confusion between the velocity of a point in the particle and the velocity of its centre; so we will drop the zero subscript of $\boldsymbol{v}_0$, $\boldsymbol{a}_0$ and γ_0.

4. Stresses in the charged particle

Consider a particle with a spherically symmetric charge density, that in the rest frame is $\rho(y) = qg(y)$, where q is the charge. The fraction $Q(r)$ of charge contained inside a sphere of radius r is defined as

$$Q(r) = 4\pi \int_0^r \mathrm{d}y\, y^2 g(y); \tag{20}$$

of course

$$\frac{\mathrm{d}Q}{\mathrm{d}y} = 4\pi y^2 g(y) \tag{21}$$

and $g(y)$ is normalized to 1,

$$\int \mathrm{d}^3 y\, g(y) = \lim_{y\to\infty} Q(y) = 1. \tag{22}$$

The convergence of the integral requires

$$\lim_{y\to\infty} y^3 g(y) = 0. \tag{23}$$

It is also required that the electrostatic energy be finite, which implies

$$\lim_{y\to 0} y^{5/2} g(y) = 0 \tag{24}$$

and

$$\lim_{y\to 0} y^{-1/2} Q(y) = 0. \tag{25}$$

The force density is the sum of an external force density and the contribution of its own electromagnetic fields, $\boldsymbol{f} = \boldsymbol{f}_{\mathrm{ex}} + \boldsymbol{f}_{\mathrm{em}}$, with $\boldsymbol{f}_{\mathrm{em}} = \rho\boldsymbol{E} + c^{-1}\boldsymbol{j}\times\boldsymbol{B}$. The electromagnetic field $F^{\mu\nu}$ has two components, a local field $F_{\mathrm{L}}^{\mu\nu}$, which decays as r^{-2}, and the radiation field $F_{\mathrm{R}}^{\mu\nu}$, which decays as r^{-1} and is proportional to the acceleration. The self-force $\boldsymbol{f}_{\mathrm{em}}$ therefore has four terms corresponding to the electric and magnetic forces of the local and radiation fields. The local electric field depends on acceleration $\boldsymbol{E}_{\mathrm{L}}(\boldsymbol{a})$. We will define the electrostatic force density as $\boldsymbol{f}_{\mathrm{es}} = \rho\boldsymbol{E}_{\mathrm{L}}(0)$; this is the largest term; by definition it is independent of acceleration, but by symmetry it does not contribute to the net force. It does contribute to the power. The remaining local electric force density $\rho(\boldsymbol{E}_{\mathrm{L}}(\boldsymbol{a}) - \boldsymbol{E}_{\mathrm{L}}(0))$ and the other three terms will be included in the field reaction force density $\boldsymbol{f}_{\mathrm{fr}}$, so $\boldsymbol{f}_{\mathrm{em}} = \boldsymbol{f}_{\mathrm{es}} + \boldsymbol{f}_{\mathrm{fr}}$. The force of the local magnetic field, of course does not contribute to the power, but it does give a contribution to the total force, which is of the first order in the acceleration.

The momentum and energy of the local field are bound to the particle and contribute to its total energy and momentum. When the particle moves with constant velocity the local field is easily obtained with a Lorentz transformation of the electric field at rest. When the particle is accelerated the local field is modified far away from the particle. Hence, the local field momentum and energy are only slightly dependent on the acceleration. The zeroth order can be calculated using the constant velocity fields.

At rest the electric field is

$$\boldsymbol{E} = q\frac{Q(y)}{y^2}\hat{\boldsymbol{y}}. \tag{26}$$

The electrostatic energy U_{e} can be evaluated with any of the following equivalent expressions

$$U_{\mathrm{e}} = \frac{q^2}{8\pi}\int \mathrm{d}^3 y\, \frac{Q(y)^2}{y^4} \tag{27}$$

$$= q^2 \int_0^\infty \mathrm{d}y \frac{\mathrm{d}Q}{\mathrm{d}y} \frac{Q(y)}{y} \tag{28}$$

$$= \frac{q^2}{2} \int\int \mathrm{d}^3y\, \mathrm{d}^3y' \, \frac{g(y)g(y')}{|\boldsymbol{y} - \boldsymbol{y}'|}. \tag{29}$$

At rest, (8) reduces to $\nabla P = \boldsymbol{f}_{\mathrm{es}}$, from where the pressure can be obtained. In general, it also depends on $\boldsymbol{f}_{\mathrm{ex}}$ and on $\boldsymbol{f}_{\mathrm{fr}}$, but as $f_{\mathrm{fr}} \ll f_{\mathrm{es}}$ and as it is supposed that the external force does not deform the particle, these dependencies will be neglected. In the frame of instantaneous rest, the pressure, up to the zeroth order in the acceleration, is then

$$P(r) = -q^2 \int_r^\infty \mathrm{d}y \, \frac{g(y)Q(y)}{y^2} + \mathrm{O}(a). \tag{30}$$

By integration of this expression over the volume, the pressure mass is obtained,

$$m_{\mathrm{P}} = -\frac{U_{\mathrm{e}}}{3c^2} + \mathrm{O}(a). \tag{31}$$

When the particle moves with constant velocity the fields are

$$\boldsymbol{E}_{\mathrm{L}} = q\gamma \frac{Q(y)}{y^3} \boldsymbol{r} \tag{32}$$

and

$$\boldsymbol{B}_{\mathrm{L}} = \frac{1}{c} \boldsymbol{v} \times \boldsymbol{E}_{\mathrm{L}}, \tag{33}$$

where $\boldsymbol{r}$ is the position from the centre of the particle in the laboratory frame and $\boldsymbol{y}$ has the same meaning in the instantaneous rest frame,

$$\boldsymbol{y} = \boldsymbol{r} + (\gamma - 1)(\boldsymbol{r} \cdot \hat{\boldsymbol{v}})\hat{\boldsymbol{v}}. \tag{34}$$

The local field energy and momentum are calculated integrating, over the volume, the energy density $u_{\mathrm{L}} = (E_{\mathrm{L}}^2 + B_{\mathrm{L}}^2)/(8\pi)$ and the momentum density $c^{-2}\boldsymbol{S}_{\mathrm{L}}, \boldsymbol{S}_{\mathrm{L}} = c\boldsymbol{E}_{\mathrm{L}} \times \boldsymbol{B}_{\mathrm{L}}/(4\pi)$. The results are

$$U_{\mathrm{L}} = U_{\mathrm{e}}\gamma \left(1 + \frac{1}{3}\left(\frac{v}{c}\right)^2\right) + \mathrm{O}(a) \tag{35}$$

and

$$\boldsymbol{P}_{\mathrm{L}} = \frac{4}{3} c^{-2} U_{\mathrm{e}} \gamma \boldsymbol{v} + \mathrm{O}(a). \tag{36}$$

If we add these results to the energy $(m_0 + m_{\mathrm{P}}(1-\gamma^{-2}))c^2\gamma$ and momentum $(m_0 + m_{\mathrm{P}})\gamma\boldsymbol{v}$ of the particle, we obtain the expected results for the dressed particle, $(m_0c^2 + U_{\mathrm{e}})\gamma$ and $(m_0 + U_{\mathrm{e}}/c^2)\gamma\boldsymbol{v}$.

Why does the momentum of the field have the famous $4/3$ factor? Exactly for the same reason why the momentum of the particle is not the bare one: because the field is not free and therefore it has stress. In the particle case, the stress tensor $P^{\mu\nu}$ is well separated from the energy–momentum tensor $\Theta^{\mu\nu}$. The volume integrals of the time components of the energy–momentum tensor transform as a four-vector. The volume integrals of the time components of the stress tensor, that is the energy and momentum contributions of stress, do not transform as a four-vector. In the field case, it is not possible in general to split the tensor $T^{\mu\nu}$ into an energy–momentum tensor plus a stress tensor. So the volume integrals of $T^{\mu 0}$ do not transform as a four-vector. That is, the total energy and momentum of the field do not form a four-vector. There is nothing wrong with this. Relativity only requires that the *total* energy and momentum of a *confined* system should form a four-vector. Indeed the effect of the stress of the field is cancelled by the effect of the stress of the matter. In the particular case of a single rigid charged body moving with constant velocity, it is possible to obtain separated stress and energy–momentum tensors of the field. In the rest frame, T^{00} is the energy–momentum tensor while the stress is T^{ij}. All the above discussion is consistent with the results of Singal [11].

The stress tensor $P^{\mu\nu}$ given by (7) and (30) is identical to the term that Schwinger [10] found that had to be added to $T^{\mu\nu}$ in order to obtain a four-momentum with electromagnetic mass U_e/c^2. We see that no ad hoc assumptions are needed, the term is just the pressure inside the particle.

In order to completely solve the 4/3 problem, we have to prove that (19) is verified. The electrostatic force density is

$$\boldsymbol{f}_{\text{es}} = q^2\gamma^2 g(y)\frac{Q(y)}{y^3}\boldsymbol{r}. \tag{37}$$

It is obvious that $\boldsymbol{f}_{\text{es}}$ does not contribute to $\boldsymbol{F}_{\text{em}}$ because $\int \mathrm{d}^3x\, \boldsymbol{f}_{\text{es}} = 0$, but it is the leading contribution to the power unbalance. In fact, we are assuming that $\boldsymbol{f}_{\text{ex}}$ does not stress the particle, so $\int \mathrm{d}^3x\, \boldsymbol{f}_{\text{ex}} \cdot \delta\boldsymbol{v} = 0$. On the other hand, the electric component of the field reaction force density $\boldsymbol{f}_{\text{fr}}$ is proportional to the acceleration, therefore it gives a higher-order contribution to the power integral. Therefore,

$$\int \mathrm{d}^3x\, \boldsymbol{f}\cdot\delta\boldsymbol{v} = \int \mathrm{d}^3x\, \boldsymbol{f}_{\text{es}}\cdot\delta\boldsymbol{v} + \mathrm{O}(a^2). \tag{38}$$

To calculate this integral we note that

$$\delta\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \tag{39}$$

and that $r^2 = y^2 + (\gamma^{-2} - 1)(\boldsymbol{y}\cdot\hat{\boldsymbol{v}})^2$.

$$\int \mathrm{d}^3x\, \boldsymbol{f}_{\text{es}}\cdot\delta\boldsymbol{v} = \frac{q^2\gamma^2}{2}\int \mathrm{d}^3x\, g(y)\frac{Q(y)}{y^3}\frac{\mathrm{d}r^2}{\mathrm{d}t} \tag{40}$$

$$= \frac{q^2\gamma}{2}\int \mathrm{d}^3y\, g(y)\frac{Q(y)}{y^3}\frac{\mathrm{d}}{\mathrm{d}t}[(\gamma^{-2} - 1)(\boldsymbol{y}\cdot\hat{\boldsymbol{v}})^2] \tag{41}$$

$$= \frac{q^2\gamma}{6}\int \mathrm{d}y\, \frac{\mathrm{d}Q}{\mathrm{d}y}\frac{Q(y)}{y}\frac{\mathrm{d}\gamma^{-2}}{\mathrm{d}t} \tag{42}$$

$$= \frac{1}{3}U_e\frac{\mathrm{d}\gamma^{-1}}{\mathrm{d}t} \tag{43}$$

as expected from (19).

5. Conservation laws of the fields

In this section, we will consider the conservation laws of energy and momentum of the fields. If $T^{\mu\nu}$ is the momentum–energy–stress tensor of the electromagnetic field and f^{μ}_{em} is the force density acting on the particle, then the conservation law is

$$\nabla_\mu T^{\mu\nu} = -f^{\nu}_{\text{em}}. \tag{44}$$

As the electromagnetic field is the sum of local and radiation terms, the tensor, which is quadratic, contains cross-terms that are products of local and radiation fields,

$T^{\mu\nu} = T_{\rm L}^{\mu\nu} + T_{\rm C}^{\mu\nu} + T_{\rm R}^{\mu\nu}$. Introducing energy densities, Poynting vectors and Maxwell tensors, (44) is equivalent to

$$\frac{\partial}{\partial t}(u_{\rm L} + u_{\rm C} + u_{\rm R}) + \nabla\cdot(\boldsymbol{S}_{\rm L} + \boldsymbol{S}_{\rm C} + \boldsymbol{S}_{\rm R}) = -\boldsymbol{f}_{\rm em}\cdot(\boldsymbol{v} + \delta\boldsymbol{v}) \tag{45}$$

and

$$\frac{\partial}{c^2\partial t}(\boldsymbol{S}_{\rm L} + \boldsymbol{S}_{\rm C} + \boldsymbol{S}_{\rm R}) + \nabla\cdot(\mathbf{T}_{\rm L} + \mathbf{T}_{\rm C} + \mathbf{T}_{\rm R}) = -\boldsymbol{f}_{\rm em}. \tag{46}$$

Now we will integrate both equations over the whole space. One has to be careful because $T_{\rm R}^{\mu\nu}$ decays as r^{-2}. So we take a volume V, surrounding the particle, so large that $T_{\rm L}^{\mu\nu} \approx 0$ and $T_{\rm C}^{\mu\nu} \approx 0$ outside V. In these conditions, the radiated power and the rate of radiated momentum defined as follows are independent of V.

$$P_{\rm rad} = \frac{\rm d}{{\rm d}t}\int_V {\rm d}^3x\, u_{\rm R} + \int_{\partial V} {\rm d}\boldsymbol{A}\cdot\boldsymbol{S}_{\rm R}, \tag{47}$$

$$\boldsymbol{G}_{\rm rad} = \frac{\rm d}{c^2{\rm d}t}\int_V {\rm d}^3x\, \boldsymbol{S}_{\rm R} + \int_{\partial V} {\rm d}\boldsymbol{A}\cdot\mathbf{T}_{\rm R}. \tag{48}$$

We get

$$\frac{\rm d}{{\rm d}t}(U_{\rm L} + U_{\rm C}) + P_{\rm rad} = -\boldsymbol{F}_{\rm em}\cdot\boldsymbol{v} - \frac{U_{\rm e}}{3}\frac{{\rm d}\gamma^{-1}}{{\rm d}t} \tag{49}$$

and

$$\frac{\rm d}{{\rm d}t}(\boldsymbol{P}_{\rm L} + \boldsymbol{P}_{\rm C}) + \boldsymbol{G}_{\rm rad} = -\boldsymbol{F}_{\rm em}. \tag{50}$$

Here, we have used (40) and the total cross-energy and momentum defined as

$$U_{\rm C} = \int {\rm d}^3x\, u_{\rm C} \tag{51}$$

and

$$\boldsymbol{P}_{\rm C} = \frac{1}{c^2}\int {\rm d}^3x\, \boldsymbol{S}_{\rm C}. \tag{52}$$

The dressed particle includes, in addition to the matter momentum and the pressure momentum, the momentum of the local fields. The reaction of these is $\boldsymbol{F}_{\rm L} = -\dot{\boldsymbol{P}}_{\rm L}$. This reaction should be subtracted from the self-force $\boldsymbol{F}_{\rm em}$ in order to obtain the force that the dressed particle feels. We will call this last force radiation reaction,

$$\boldsymbol{F}_{\rm R} = \boldsymbol{F}_{\rm em} - \boldsymbol{F}_{\rm L} = \boldsymbol{F}_{\rm em} + \frac{{\rm d}\boldsymbol{P}_{\rm L}}{{\rm d}t}. \tag{53}$$

Actually, $\boldsymbol{F}_{\rm R}$ contains not only the reaction of the radiated fields but also the reaction of the cross-terms,

$$\boldsymbol{F}_{\rm R} = -\boldsymbol{G}_{\rm rad} - \frac{{\rm d}\boldsymbol{P}_{\rm C}}{{\rm d}t}. \tag{54}$$

The power of $\boldsymbol{F}_{\rm R}$ is obtained from (49) using (35), (36) and the identity (18). It also contains a contribution from the cross-terms,

$$\boldsymbol{F}_{\rm R}\cdot\boldsymbol{v} = -P_{\rm rad} - \frac{{\rm d}U_{\rm C}}{{\rm d}t}. \tag{55}$$

Finally, the motion equation and the power equation of the dressed particle are readily obtained from (14) and (17), with dressed mass $m = m_0 + U_{\rm e}/c^2$

$$m\frac{{\rm d}\gamma\boldsymbol{v}}{{\rm d}t} = \boldsymbol{F}_{\rm ex} + \boldsymbol{F}_{\rm R}, \tag{56}$$

$$mc^2\frac{{\rm d}\gamma}{{\rm d}t} = (\boldsymbol{F}_{\rm ex} + \boldsymbol{F}_{\rm R})\cdot\boldsymbol{v}. \tag{57}$$

6. Radiation reaction

The calculation of the self-force is much simpler if it is done in the reference frame of instantaneous rest. To begin with, there are no magnetic field contributions so we have only to calculate the electric field. It is very convenient in this case to split the electric field into electrostatic and induced components, $\boldsymbol{E} = \boldsymbol{E}_{\mathrm{e}} + \boldsymbol{E}_{\mathrm{i}}$, with the following properties: $\nabla \cdot \boldsymbol{E}_{\mathrm{e}} = 4\pi\rho$, $\nabla \times \boldsymbol{E}_{\mathrm{e}} = 0$ and $\nabla \cdot \boldsymbol{E}_{\mathrm{i}} = 0$. Note that $\boldsymbol{E}_{\mathrm{e}}$ is the longitudinal and $\boldsymbol{E}_{\mathrm{i}}$ the transverse field. With these definitions, in the rest frame, $\boldsymbol{f}_{\mathrm{es}} = \rho\boldsymbol{E}_{\mathrm{e}}$ and $\boldsymbol{f}_{\mathrm{fr}} = \rho\boldsymbol{E}_{\mathrm{i}}$. As we have previously explained, in the integrated self-force $\boldsymbol{F}_{\mathrm{em}}$ the electrostatic contribution cancels out. The easiest way to determine these fields is to use the Coulomb gauge, $\nabla \cdot \boldsymbol{A} = 0$, because in this case the potential ϕ yields $\boldsymbol{E}_{\mathrm{e}}$ and $\boldsymbol{A}$ yields $\boldsymbol{E}_{\mathrm{i}}$,

$$\boldsymbol{E}_{\mathrm{e}} = -\nabla\phi \tag{58}$$

and

$$\boldsymbol{E}_{\mathrm{i}} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}. \tag{59}$$

The electrostatic field is

$$\boldsymbol{E}_{\mathrm{e}}(\boldsymbol{x}, t) = \int \mathrm{d}^3 y \frac{\rho(\boldsymbol{y}, t)}{|\boldsymbol{x} - \boldsymbol{y}|^3}(\boldsymbol{x} - \boldsymbol{y}) \tag{60}$$

and $\boldsymbol{A}$ can be obtained from the equation

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}_{\mathrm{e}}}{\partial t}. \tag{61}$$

The solution is

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{c}\int \mathrm{d}^3 y \frac{\boldsymbol{j}(\boldsymbol{y}, t')}{|\boldsymbol{x} - \boldsymbol{y}|} + \frac{1}{4\pi c}\int \mathrm{d}^3 y \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\frac{\partial \boldsymbol{E}_{\mathrm{e}}(\boldsymbol{y}, t')}{\partial t}, \tag{62}$$

where t' is the retarded time, $t' = t - |\boldsymbol{x} - \boldsymbol{y}|/c$. The induced field is therefore

$$\boldsymbol{E}_{\mathrm{i}}(\boldsymbol{x}, t) = -\frac{1}{c^2}\int \mathrm{d}^3 y \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\frac{\partial \boldsymbol{j}(\boldsymbol{y}, t')}{\partial t} - \frac{1}{4\pi c^2}\int \mathrm{d}^3 y \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\frac{\partial^2 \boldsymbol{E}_{\mathrm{e}}(\boldsymbol{y}, t')}{\partial t^2}. \tag{63}$$

For the quasi-rigid motion the charge density is $\rho(\boldsymbol{x}, t) = q\gamma g(y)$ and the current density is $\boldsymbol{j} = \rho(\boldsymbol{v} + \delta\boldsymbol{v})$, where $\boldsymbol{y} = \boldsymbol{r} + (\gamma - 1)(\boldsymbol{r} \cdot \hat{\boldsymbol{v}})\hat{\boldsymbol{v}}$, $y = |\boldsymbol{y}|$ and $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0(t)$. With these definitions we have in the rest frame

$$\frac{\partial^2 \rho(\boldsymbol{x}, t)}{\partial t^2} = -q\frac{\mathrm{d}g}{\mathrm{d}r}\hat{\boldsymbol{r}} \cdot \boldsymbol{a}(t) + \mathrm{O}(a^2) \tag{64}$$

and

$$\frac{\partial \boldsymbol{j}(\boldsymbol{x}, t)}{\partial t} = qg(r)\boldsymbol{a}(t) + \mathrm{O}(a^2). \tag{65}$$

If one now replaces the expression for ρ into the electrostatic field equation one gets

$$\frac{\partial^2 \boldsymbol{E}_{\mathrm{e}}(\boldsymbol{x}, t)}{\partial t^2} = q\frac{\partial}{\partial \boldsymbol{x}}\int d^3 y \frac{\mathrm{d}g}{\mathrm{d}y}\frac{\hat{\boldsymbol{y}} \cdot \boldsymbol{a}(t)}{|\boldsymbol{x} - \boldsymbol{y}|}. \tag{66}$$

The integral in (66) can be evaluated as

$$\int \mathrm{d}^3 y \frac{\mathrm{d}g}{\mathrm{d}y}\frac{\hat{\boldsymbol{y}}}{|\boldsymbol{x} - \boldsymbol{y}|} = -\frac{Q(x)}{x^2}\hat{\boldsymbol{x}}, \tag{67}$$

here $Q(x)$ is defined in (20) and we have used the fact that

$$\frac{1}{4\pi}\int \mathrm{d}\Omega_y \frac{\hat{\boldsymbol{y}}}{|\boldsymbol{x} - \boldsymbol{y}|} = \frac{\min(x, y)}{3\max(x, y)^2}\hat{\boldsymbol{x}}. \tag{68}$$

The second derivative of the electrostatic field is obtained by taking the gradient of (67), then

$$\frac{\partial^2 \boldsymbol{E}_{\mathrm{e}}(\boldsymbol{x}, t)}{\partial t^2} = -q\left[4\pi g(x)\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \frac{Q(x)}{x^3}(\mathbf{I} - 3\hat{\boldsymbol{x}}\hat{\boldsymbol{x}})\right] \cdot \boldsymbol{a}(t). \tag{69}$$

Here, $\mathbf{I}$ is the identity tensor. This expression should be replaced into (63) and then integrated again in order to obtain the self-force

$$\boldsymbol{F}_{\mathrm{em}}(t) = q \int \mathrm{d}^3x\, g(x)\boldsymbol{E}_{\mathrm{i}}(\boldsymbol{x}, t). \tag{70}$$

In this process, the induced field is spherically averaged. Because of (65) and as

$$\frac{1}{4\pi}\int \mathrm{d}\Omega_x\, \hat{\boldsymbol{x}}\hat{\boldsymbol{x}} = \frac{1}{3}\mathbf{I}, \tag{71}$$

one finds that the contribution of the second term of (63) is exactly $-1/3$ of that of the first term. Finally, we obtain the self-force formula in the rest frame

$$\boldsymbol{F}_{\mathrm{em}}(t) = -\frac{2}{3}\frac{q^2}{c^2}\int\int \mathrm{d}^3x\, \mathrm{d}^3y \frac{g(x)g(y)}{|\boldsymbol{x} - \boldsymbol{y}|}\boldsymbol{a}(t - |\boldsymbol{x} - \boldsymbol{y}|/c). \tag{72}$$

As was expected it is an average of the retarded accelerations, therefore no violation of causality is due to this formula. By using the Coulomb gauge we have obtained $\boldsymbol{F}_{\mathrm{em}}$ in closed form instead of the troublesome power expansion one gets when the Lorentz gauge is used. The sum of that power expansion appears in the textbook of Jackson [3], and indeed our equation (72) can be deduced from equations (16.28)–(16.30) of that book.

When the acceleration is constant the self-force reduces to the reaction of the local fields,

$$\boldsymbol{F}_{\mathrm{L}}(t) = -\frac{4}{3}\frac{U_{\mathrm{e}}}{c^2}\boldsymbol{a}(t). \tag{73}$$

One has to subtract this contribution in order to obtain the radiation reaction force

$$\boldsymbol{F}_{\mathrm{R}}(t) = \frac{2}{3}\frac{q^2}{c^2}\int\int \mathrm{d}^3x\, \mathrm{d}^3y \frac{g(x)g(y)}{|\boldsymbol{x} - \boldsymbol{y}|}\big(\boldsymbol{a}(t) - \boldsymbol{a}(t - |\boldsymbol{x} - \boldsymbol{y}|/c)\big). \tag{74}$$

If one expands the retarded acceleration in terms of $|\boldsymbol{x} - \boldsymbol{y}|$ the LAD result is recovered as the first term in the expansion

$$\boldsymbol{F}_{\mathrm{R}} = \frac{2}{3}\frac{q^2}{c^3}\dot{\boldsymbol{a}}(t) + \mathrm{O}(R). \tag{75}$$

If one uses (72) to calculate the self-force of a charged spherical shell of radius R, the Sommerfeld–Page–Caldirola–Yaghjian result is obtained,

$$\boldsymbol{F}_{\mathrm{em}}(t) = -\frac{1}{3}\frac{q^2}{cR^2}(\boldsymbol{v}(t) - \boldsymbol{v}(t - 2R/c)). \tag{76}$$

7. Integral equation and point particle equation

In the rest frame, the equation of motion is

$$m\boldsymbol{a} = \boldsymbol{F}_{\mathrm{ex}} + \boldsymbol{F}_{\mathrm{R}}. \tag{77}$$

This is, of course, the non-relativistic equation. The condition $a \ll c^2/2R$ implicit in the quasi-rigid approximation implies that $v \ll c$ for times of the order of the transit time (i.e. the time that light takes to traverse the particle) and therefore the non-relativistic equation can be used. Our purpose is to solve (77) in order to express the radiation reaction as a functional of the external force, instead of a functional of the acceleration. Equation (74) can be written as

$$\boldsymbol{F}_{\mathrm{R}}(t) = \int_0^\infty \mathrm{d}t'\, k(t')\big(\boldsymbol{a}(t) - \boldsymbol{a}(t-t')\big), \tag{78}$$

where the kernel k is given by

$$k(t) = \frac{2q^2}{3c^2}\int\int \mathrm{d}^3x\,\mathrm{d}^3y\,\frac{g(x)g(y)}{|\boldsymbol{x}-\boldsymbol{y}|}\delta(t - |\boldsymbol{x}-\boldsymbol{y}|/c). \tag{79}$$

The solution is obtained by means of a Fourier transform

$$K(\omega) = \int_0^\infty \mathrm{d}t\, k(t)\,\mathrm{e}^{\mathrm{i}\omega t} = \frac{2q^2}{3c^2}\int\int \mathrm{d}^3x\,\mathrm{d}^3y\,\frac{g(x)g(y)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{e}^{\mathrm{i}\omega|\boldsymbol{x}-\boldsymbol{y}|/c}. \tag{80}$$

The mass of the local fields equals $K(0)$,

$$K(0) = \int_0^\infty \mathrm{d}t\, k(t) = \frac{2}{3}\frac{q^2}{c^2}\int\int \mathrm{d}^3x\,\mathrm{d}^3y\,\frac{g(x)g(y)}{|\boldsymbol{x}-\boldsymbol{y}|} = \frac{4}{3c^2}U_{\mathrm{e}}, \tag{81}$$

while the derivative of $K(\omega)$ at $\omega = 0$ gives the constant of the LAD equation,

$$-\mathrm{i}\frac{\mathrm{d}K}{\mathrm{d}\omega}\bigg|_0 = \int_0^\infty \mathrm{d}t\, tk(t) = \frac{2}{3}\frac{q^2}{c^3}\int\int \mathrm{d}^3x\,\mathrm{d}^3y\, g(x)g(y) = \frac{2}{3}\frac{q^2}{c^3}. \tag{82}$$

From (81) and (82), one obtains an expression for the average delay time τ

$$\tau = \frac{1}{K(0)}\int_0^\infty \mathrm{d}t\, tk(t) = \left(c\int\int \mathrm{d}^3x\,\mathrm{d}^3y\,\frac{g(x)g(y)}{|\boldsymbol{x}-\boldsymbol{y}|}\right)^{-1}. \tag{83}$$

The acceleration, obtained from (77), is then

$$\boldsymbol{a}(t) = \int_{-\infty}^\infty \mathrm{d}t'\, G(t')\boldsymbol{F}_{\mathrm{ex}}(t-t'), \tag{84}$$

where

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}\omega\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{m - K(0) + K(\omega)}. \tag{85}$$

Causality requires that $G(t) = 0$ for $t < 0$. This implies that $(m - K(0) + K(\omega))^{-1}$ must be regular for $\mathrm{Im}(\omega) > 0$. $K(\omega)$ is regular for $\mathrm{Im}(\omega) > 0$, so the only possible singularities are poles when $m - K(0) + K(\omega) = 0$. For $\eta \geqslant 0$, $K(\mathrm{i}\eta)$ is real and $\lim_{\eta\to\infty} K(\mathrm{i}\eta) = 0$. When $g(r) \geqslant 0$, $K(\mathrm{i}\eta)$ decreases monotonically from $K(0)$ to zero. Even when there are regions in which $g(r)$ is negative $K(\mathrm{i}\eta) > 0$. Therefore, $(m - K(0) + K(\omega))^{-1}$ is regular in the upper half of the complex plane if and only if, $m > K(0)$. That is, it must be $m > \frac{4}{3}U_{\mathrm{e}}/c^2$, or equivalently $m_0 > \frac{1}{3}U_{\mathrm{e}}/c^2$. Note that this is the same condition discussed in section 3. If the condition is fulfilled $G(t) \sim \theta(t)\exp(-t/\tau_G)$ and $\tau_G \sim \tau$. If the condition is not satisfied, causality is violated and run-away solutions appear. Let us look in more detail how this happens. Consider a case in which the external force was constant for $t < 0$, and then suddenly vanishes at $t = 0$. The self-force $\boldsymbol{F}_{\mathrm{em}}$ given by (72) is opposite to the acceleration and remains the same at the transition $\boldsymbol{F}_{\mathrm{em}}(0^+) = \boldsymbol{F}_{\mathrm{em}}(0^-) = -K(0)\boldsymbol{a}(0^-)$. The motion is determined by (14). For $t > 0$, the acceleration equals $\boldsymbol{F}_{\mathrm{em}}$ divided by the effective mass of matter $m_0 + m_{\mathrm{P}} = m - K(0)$, in particular $\boldsymbol{a}(0^+) = -(m - K(0))^{-1}K(0)\boldsymbol{a}(0^-)$. If $m_0 + m_{\mathrm{P}} > 0$ the new acceleration is opposite to that of negative times and, as time elapses, the accelerations for $t > 0$ subtract from the previous ones in (72); therefore the self-force is reduced and the acceleration vanishes exponentially. On the contrary, if $0 < m < K(0)$, the new acceleration not only has the same sense of the previous one but it is actually larger than it. As time elapses the self-force increases and the acceleration diverges exponentially.

The mass condition was previously demonstrated for the spherical charged shell by Moniz and Sharp [19]. Although their quantum-mechanical results have been disputed [20], their classical calculation of the run-away solutions is correct, apart from the fact that they do not include the pressure contribution to the mechanical mass. Here, we have shown that the mass condition is valid for any charge distribution and have clarified its relation with the inertia of stress.

The equation (84) can be re-written as

$$\boldsymbol{a}(t) = \left[\int_0^\infty \mathrm{d}t'\, G(t')\right]\boldsymbol{F}_{\mathrm{ex}}(t) + \int_0^\infty \mathrm{d}t'\, G(t')(\boldsymbol{F}_{\mathrm{ex}}(t-t') - \boldsymbol{F}_{\mathrm{ex}}(t)). \tag{86}$$

The pre-factor of the force in the first term of (86) happens to be m^{-1}

$$\int_0^\infty \mathrm{d}t\, G(t) = \int_{-\infty}^\infty \mathrm{d}\omega \frac{\delta(\omega)}{m - K(0) + K(\omega)} = \frac{1}{m}. \tag{87}$$

The radiation reaction is then

$$\boldsymbol{F}_{\mathrm{R}} = m\int_0^\infty \mathrm{d}t'\, G(t')(\boldsymbol{F}_{\mathrm{ex}}(t-t') - \boldsymbol{F}_{\mathrm{ex}}(t)). \tag{88}$$

Finally, if the external force varies slowly enough, more precisely if

$$\frac{|\dot{F}_{\mathrm{ex}}|}{F_{\mathrm{ex}}} \ll \tau^{-1}, \tag{89}$$

an approximated instantaneous radiation reaction is obtained by taking the first term of the expansion in t' of the retarded force in (88),

$$\boldsymbol{F}_{\mathrm{R}} = \frac{2}{3}\frac{q^2}{c^3 m}\dot{\boldsymbol{F}}_{\mathrm{ex}}. \tag{90}$$

To obtain this we have used the fact that

$$\begin{aligned}\int_0^\infty \mathrm{d}t\, tG(t) &= \mathrm{i}\int_{-\infty}^\infty \mathrm{d}\omega \frac{1}{m - K(0) + K(\omega)}\frac{\mathrm{d}\delta(\omega)}{\mathrm{d}\omega} \\ &= \frac{\mathrm{i}}{m^2}\frac{\mathrm{d}K}{\mathrm{d}\omega}\bigg|_0 = -\frac{2}{3}\frac{q^2}{c^3 m^2}.\end{aligned} \tag{91}$$

Equation (90) was proposed by Rohrlich [17] as the correct one for point particles. We see that as long as the mass condition $m > \frac{4}{3}U_{\mathrm{e}}/c^2$ is fulfilled, the radiation reaction can be expressed by (88), which conforms to causality and does not produce run-away solutions. The approximated expression of (90) does produce violations of causality when $F_{\mathrm{R}} \sim F_{\mathrm{ex}}$. When this happens the condition of (89) is not satisfied, so the approximation is not valid.

A particle can be considered a point particle if its radius is negligible when compared with the wavelength of the radiated field. This condition is equivalent to (89). Therefore, (90) can indeed be considered the correct one for point particles. It is worth noting that applied forces with discontinuities that violate (89), are very common. However, usually such discontinuities are not real, but the result of some approximation. In most cases the formula of (90) behaves properly if, instead of discontinuities, realistic smoothed steps are used.

8. Radiation reaction four-force and radiated power

The radiation reaction of the physical point charge with $\boldsymbol{v} = 0$ is given by equation (90). In this section, we find this force for a particle moving with arbitrary speed.

A four-force K^μ that conserves mass is related to the force by $K^i = \gamma F^i$ and to the power by $K^0 = \gamma \boldsymbol{F} \cdot \boldsymbol{v}/c$. This implies that it is orthogonal to the four-velocity $K^\mu u_\mu = 0$. In the rest frame, $K^i = F^i$ and $K^0 = 0$.

The motion equations of the dressed mass, (56) and (57), show that the radiation reaction is a mass-conserving force. Then, the four-force of the radiation reaction of the physical point charge is in the rest frame

$$K_{\mathrm{R}}^i = \frac{2}{3}\frac{q^2}{c^3 m}\frac{\mathrm{d}F_{\mathrm{ex}}^i}{\mathrm{d}t} \tag{92}$$

and

$$K_{\mathrm{R}}^0 = 0. \tag{93}$$

Therefore, if we call K^μ the external four-force and τ the proper time, the radiation reaction four-force is in an arbitrary reference frame given by

$$K_{\mathrm{R}}^\mu = \frac{2}{3}\frac{q^2}{c^3 m}\left[\frac{\mathrm{d}K^\mu}{\mathrm{d}\tau} + \frac{1}{c^2}\left(\frac{\mathrm{d}K^\nu}{\mathrm{d}\tau}u_\nu\right)u^\mu\right]. \tag{94}$$

The four-scalar in parenthesis is

$$\frac{\mathrm{d}K^\mu}{\mathrm{d}\tau}u_\mu = -\gamma^3 \boldsymbol{F}_{\mathrm{ex}} \cdot \boldsymbol{a}. \tag{95}$$

From (94), the force and power expressions are obtained for any speed,

$$\boldsymbol{F}_{\mathrm{R}} = \frac{2}{3}\frac{q^2}{c^3 m}\left[\frac{\mathrm{d}\gamma \boldsymbol{F}_{\mathrm{ex}}}{\mathrm{d}t} - \frac{\gamma^3}{c^2}(\boldsymbol{F}_{\mathrm{ex}} \cdot \boldsymbol{a})\boldsymbol{v}\right] \tag{96}$$

and

$$\boldsymbol{F}_{\mathrm{R}} \cdot \boldsymbol{v} = \frac{2}{3}\frac{q^2}{c^3 m}\left[\frac{\mathrm{d}\gamma \boldsymbol{F}_{\mathrm{ex}} \cdot \boldsymbol{v}}{\mathrm{d}t} - \gamma^3 \boldsymbol{F}_{\mathrm{ex}} \cdot \boldsymbol{a}\right]. \tag{97}$$

From these equations, and equations (54) and (55), one obtains the momentum and energy of the cross-terms, the radiated power and rate of radiated momentum,

$$\boldsymbol{P}_{\mathrm{C}} = -\frac{2}{3}\frac{q^2}{c^3 m}\gamma \boldsymbol{F}_{\mathrm{ex}}, \tag{98}$$

$$U_{\mathrm{C}} = -\frac{2}{3}\frac{q^2}{c^3 m}\gamma \boldsymbol{F}_{\mathrm{ex}} \cdot \boldsymbol{v} = \boldsymbol{P}_{\mathrm{C}} \cdot \boldsymbol{v}, \tag{99}$$

$$P_{\mathrm{rad}} = \frac{2}{3}\frac{q^2}{c^3 m}\gamma^3 \boldsymbol{F}_{\mathrm{ex}} \cdot \boldsymbol{a}, \tag{100}$$

$$\boldsymbol{G}_{\mathrm{rad}} = \frac{2}{3}\frac{q^2}{c^3 m}\frac{\gamma^3}{c^2}(\boldsymbol{F}_{\mathrm{ex}} \cdot \boldsymbol{a})\boldsymbol{v} = \frac{1}{c^2}P_{\mathrm{rad}}\boldsymbol{v}. \tag{101}$$

Equation (100) reduces to the Larmor result [12] in the non-relativistic limit only if $\dot{\boldsymbol{F}}_{\mathrm{ex}} \cdot \boldsymbol{a} = 0$. The Larmor result is valid for a mathematical point charge ($R = 0$), not for an extended particle.

9. Conclusion

We have completely solved the self-force of the extended quasi-rigid particle. We have shown that, provided $m_0 > \frac{1}{3}U_{\mathrm{e}}/c^2$, it is a perfectly consistent classical system conforming to causality and conservation of energy and momentum. In order to obtain the correct equations, it is essential to use the *relativistic* conservation equations of energy and momentum for the continuous medium, even if $v \ll c$; only so the inertia of pressure is properly taken into account. The mathematical point charge ($R = 0$) is inconsistent with classical electrodynamics. For a given dressed mass m and charge q, the minimum radius that any particle can have is $(2q^2)/(3mc^2)$. Nevertheless, if the radius of a particle is much smaller than any other distance in the problem, the radius becomes irrelevant and the particle can be properly considered as a point charge from the physical point of view.

A comment about QED; a correct quantum theory of a system, should give in the classical limit a proper classical system. As the mathematical point charge is inconsistent in classical electrodynamics, it follows that the proper quantum theory of a structureless particle (such as the electron) should have as classical limit an *extended* classical particle. At first sight one might think that this were not impossible as the single particle picture breaks down at the Compton length scale, which is α^{-1} (≈ 137) times larger than the classical radius of the particle. The relativistic quantum non-localities produce an effective particle extension. Nevertheless, the QED calculation for spin $\frac{1}{2}$ of Low [21] shows that these effects are not enough to eliminate the run-away solutions, but that the size bound is reduced by a factor of $\exp(-\alpha^{-1})$.

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